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Absence of Long-Range Order in One-Dimensional Spin Systems

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For a one-dimensional Ising model with interaction energy

$$E\{\mu\} = -\sum_{1 \leqslant i < j \leqslant N} J(j-i)\mu_i\mu_j \qquad [J(k) \geqslant 0,\ \mu_i = \pm 1]$$

it is proved that there is no long-range order at any temperature when

$$S_N = \sum_{k=1}^{N} kJ(k) = o\big([\log N]^{1/2}\big)$$

The same result is shown to hold for the corresponding plane rotator model when

$$S_N = o([\log N/\log\log N])$$

KEY WORDS: Ising model; plane rotator model; inequalities; long-range order.

1. INTRODUCTION

Some years ago Dyson[1] investigated the existence of phase transitions in the infinite Ising ferromagnet with interaction energy

$$H = -\sum_{i<j} J(j-i)\mu_i\mu_j \qquad (\mu_i = \pm 1)$$

He showed that provided $(\forall n)\ J(n) \geqslant 0$ and

$$\lim_{N\to\infty}\left[(\log\log N)^{-1}\sum_{n=1}^{N} nJ(n)\right] = 0 \tag{1}$$

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no long-range order exists. That is

$$\lim_{N\to\infty}\left\langle\left(\frac{1}{N}\sum_{i=1}^{N}\mu_i\right)^2\right\rangle_{\text{Ising}}=0$$

where $\langle\cdot\rangle_{\text{Ising}}$ denotes the Ising model correlation function defined below. On the other hand, he proved[1] that there is long-range order at low enough temperatures if there is $k>0$ such that $(\forall n)$

$$J(n)>kn^{-2}\log\log(n+2) \tag{2}$$

Observe that the gap between (1) and (2) includes the interaction $J(n)=n^{-2}$. The behavior of this case remains undecided although heuristic arguments[2] indicate that a phase transition should occur.

Here we pursue the nonexistence question for the Ising model and for the simplest model with a continuous symmetry, the one-dimensional plane rotator model. Both are special cases of the n-vector model ($n=1$ and $n=2$, respectively) defined as follows.

Let Λ be a linear chain of N sites. At each site $i\in\Lambda$ we locate a spin $\mathbf{S}_i\in\mathbb{R}^n$ with $\|\mathbf{S}_i\|=1$. Then the phase space of the model is $Y=\mathrm{X}_{i\in\Lambda}S^{n-1}$ and the Hamiltonian of interest is

$$H(\mathbf{S}_i,J)=-\sum_{1\leqslant i<j\leqslant N}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j \tag{3}$$

We will consider a translationally invariant model $[J_{ij}=J(j-i)]$ with ferromagnetic interactions $[(\forall n)\ J(n)\geqslant 0]$.

The thermal average of $f(\mathbf{S})$ is defined by

$$\langle f(\mathbf{S})\rangle=Z_N^{-1}\int_Y f(\mathbf{S})\exp(-\beta H)\,d\mathbf{S} \tag{4}$$

where

$$Z_N=\int_Y\exp(-\beta H)\,d\mathbf{S} \tag{5}$$

and $\beta=(kT)^{-1}$ is the inverse temperature. In case $n=1$, (4) and (5) reduce to the usual sum over the configuration space $\{\mu_i=\pm 1,\ i\in\Lambda\}$.

Our aim here is to show that there is no long-range order,

$$\lim_{N\to\infty}\left\langle\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_i\right\|^2\right\rangle=0$$

if, for the Ising model ($n=1$)

$$\lim_{N\to\infty}\left[(\log N)^{-1/2}\sum_{k=1}^{N}kJ(k)\right]=0$$

and if, for the plane rotator ($n = 2$)

$$\lim_{N\to\infty}\left[\left(\frac{\log\log N}{\log N}\right)\sum_{k=1}^{N} kJ(k)\right] = 0$$

Our results rely on forming an associated n-vector model by locking together blocks of spins in the original model. We illustrate the procedure for the Ising model. The plane rotator case is left to the Appendix.

Explicitly, for the Ising model, let i, j, and $r < s$ be sites in Λ. Then

$$\begin{aligned} Z_N = &\sum_{\{\mu\,:\,\mu_r=\mu_s\}} \exp(\beta J_{rs})\exp(-\beta H^*) \\ &+ \sum_{\{\mu\,:\,\mu_r=-\mu_s\}.} \exp(-\beta J_{rs})\exp(-\beta H^*) \end{aligned}$$

and

$$\begin{aligned} Z_N\langle \mu_i\mu_j\rangle = &\sum_{\{\mu\,:\,\mu_r=\mu_s\}} \langle \mu_i\mu_j\rangle\exp(\beta J_{rs})\exp(-\beta H^*) \\ &+ \sum_{\{\mu\,:\,\mu_r=-\mu_s\}} \langle \mu_i\mu_j\rangle\exp(-\beta J_{rs})\exp(-\beta H^*) \end{aligned}$$

where

$$H^* = -\sum_{1\leqslant i<j\leqslant N} J_{ij}\mu_i\mu_j + J_{rs}\mu_r\mu_s$$

It is evident that

$$\langle \mu_i\mu_j\rangle^* = \lim_{J_{rs}\to\infty} \langle \mu_i\mu_j\rangle$$

is just the thermal average [with respect to (3) and J_{rs} set equal to 0] taken over all configurations with $\mu_r = \mu_s$. Finally, by Griffiths' second inequality,[3] which for our purposes may be written

$$\beta^{-1}\frac{\partial}{\partial J_{ij}}\langle \mathbf{S}_r\cdot\mathbf{S}_s\rangle = \langle(\mathbf{S}_r\cdot\mathbf{S}_s)(\mathbf{S}_i\cdot\mathbf{S}_j)\rangle - \langle\mathbf{S}_r\cdot\mathbf{S}_s\rangle\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle \geqslant 0 \quad (6)$$

and which is certainly valid for $n = 1$ and $n = 2$, we have

$$\langle \mu_i\mu_j\rangle \leqslant \langle \mu_i\mu_j\rangle^*$$

2. UPPER BOUND ON MEAN SQUARE MAGNETIZATION

We now return to the Hamiltonian (3) on N sites. Divide the N sites into M blocks of p sites. Lock the spins of each block by the procedure described above [i.e., successively let the interactions $(1,2)$, $(2,3), \ldots,$ $(p-1, p)$ tend to infinity and then repeat for the next block $(p+1, p+2)$, $(p+2, p+3), \ldots$].

With each block $[l]$ is associated a spin $\mathbf{T}_l \in \mathbb{R}^n$ with $\|\mathbf{T}_l\| = 1$. By our previous discussion, Griffiths' inequality (when it is valid) implies

$$\left\langle \left\| \frac{1}{N} \sum_{i=1}^{N} \mathbf{S}_i \right\|^2 \right\rangle \leqslant \left\langle \left\| \frac{1}{M} \sum_{i=1}^{M} \mathbf{T}_i \right\|^2 \right\rangle' \tag{7}$$

where $\langle\ \rangle'$ refers to the average over block configurations with respect to the Hamiltonian

$$- \sum_{1 \leqslant k < l \leqslant M} K(l-k) \mathbf{T}_k \cdot \mathbf{T}_l$$

and

$$K(l-k) = \sum_{\substack{i \in [k] \\ j \in [l]}} J(j-i)$$

Note that $K(m)$ is well defined as $J_{ij} = J(j-i)$.

Define

$$\epsilon(p) = \sum_{k=1}^{p} kJ(k)$$

Then for nearest-neighbor blocks $[k]$, $[k+1]$

$$\epsilon(p) \leqslant K(1) = \sum_{k=1}^{p} kJ(k) + \sum_{k=p+1}^{2p} (2p-k)J(k) \leqslant \epsilon(2p) \tag{8}$$

When $[k]$ and $[l]$ are not nearest neighbor blocks we have, writing $r = l - k$,

$$K(r) = \sum_{s=(r-1)p+1}^{rp} \left[s - (r-1)p \right] J(s) + \sum_{s=rp+1}^{(r+1)p-1} \left[(r+1)p - s \right] J(s)$$

A simple counting argument then shows that

$$\begin{aligned} \sum_{r=2}^{M-1} rK(r) = & \sum_{s=p+1}^{2p} 2(s-p)J(s) + \sum_{s=2p+1}^{(M-1)p} sJ(s) \\ & + \sum_{s=(M-1)p+1}^{Mp-1} (M-1)(Mp-s)J(s) \\ & \leqslant \epsilon(pM) - \epsilon(p) \end{aligned} \tag{9}$$

a result we shall use later.

Returning to the block model we have

$$Q \equiv \left\langle \left\| \frac{1}{M} \sum_{i=1}^{M} \mathbf{T}_i \right\|^2 \right\rangle' = \frac{\left\langle \left\| \frac{1}{M} \sum_{i=1}^{M} \mathbf{T}_i \right\|^2 \exp\left[\beta \sum{}^* K(l-k) \mathbf{T}_l \cdot \mathbf{T}_k \right] \right\rangle''}{\langle \exp\left[\beta \sum{}^* K(l-k) \mathbf{T}_l \cdot \mathbf{T}_k \right] \rangle''} \tag{10}$$

where $\sum^*$ represents a sum over all pairs $([k],[l])$ which are not nearest neighbor blocks and $\langle\ \rangle''$ is the thermal average with respect to the "nearest neighbor block" Hamiltonian

$$-\sum K(1) \mathbf{T}_l \cdot \mathbf{T}_{l+1}$$

Now, by Jensen's inequality

$$\langle \exp(\beta \sum{}^* K(l-k) \mathbf{T}_l \cdot \mathbf{T}_k) \rangle'' \geqslant \exp(\beta \sum{}^* K(l-k) \langle \mathbf{T}_l \cdot \mathbf{T}_k \rangle'')$$

so

$$Q \leqslant \left\langle \left\| \frac{1}{M} \sum_{i=1}^{M} \mathbf{T}_i \right\|^2 \right\rangle'' \exp\left[\beta \sum{}^* K(l-k)(1 - \langle \mathbf{T}_l \cdot \mathbf{T}_k \rangle'') \right] \tag{11}$$

The properties of the one-dimensional n-vector model are well known.[4] Thus, for finite free ends models with nearest neighbor block interaction q, we can define independently of l,

$$V(q) = \langle \mathbf{T}_l \cdot \mathbf{T}_{l+1} \rangle''$$

Then $0 \leqslant V(q) \leqslant 1$ and

$$\langle \mathbf{T}_l \cdot \mathbf{T}_k \rangle'' = \left[V(q) \right]^{|k-l|}$$

Further, by Griffiths' inequality (when it is valid), $V(q)$ is an increasing function. Hence, from (8),

$$V(\epsilon(p)) \leqslant V(q) \leqslant V(\epsilon(2p)) \tag{12}$$

It is readily verified that

$$\left\langle \left\| \frac{1}{M} \sum_{i=1}^{M} \mathbf{T}_i \right\|^2 \right\rangle'' \leqslant \frac{1}{M} \frac{2}{1 - V(q)} \leqslant \frac{2}{M W(\epsilon(2p))} \tag{13}$$

where

$$W(q) \equiv 1 - V(q) \tag{14}$$

Next, using (12) and (14) and the elementary inequality

$$(1 - W)^n \geqslant 1 - nW \qquad \text{for } W < 1$$

we obtain

$$\exp\left[\beta\sum{}^{*}K(l-k)(1-\langle \mathbf{T}_l\cdot\mathbf{T}_k\rangle'')\right]$$
$$=\exp\left(\beta\sum_{r=2}^{M-1}K(r)\sum_{i=1}^{M-r}\left\{1-\left[V(q)\right]^r\right\}\right)$$
$$\leqslant\exp\left(\beta\sum_{r=2}^{M-1}K(r)(M-r)\left\{1-\left[V(\epsilon(p))\right]^r\right\}\right)$$
$$\leqslant\exp\left[\beta M\sum_{r=2}^{M-1}K(r)rW(\epsilon(p))\right] \tag{15}$$

Hence, combining (9), (11), (13), and (15)

$$Q\leqslant\frac{2}{MW(\epsilon(2p))}\exp\left\{\beta MW(\epsilon(p))\left[\epsilon(pM)-\epsilon(p)\right]\right\} \tag{16}$$

To show that no long-range order exists in the original model it suffices by (7) and (16) to show that $Q\to 0$ as $N\to\infty$.

3. PARTICULAR MODELS

We now investigate particular n-vector models for which Griffiths' inequalities, and hence also inequality (16), are valid.

3.1. Ising Model

For $n=1$, (3) reduces to the Ising model for which

$$V(q)=\tanh\beta q=1-\frac{2e^{-2\beta q}}{1+e^{-2\beta q}}$$

An elementary calculation yields

$$\exp(-2\beta q)\leqslant 1-V(q)=W(q)\leqslant 2\exp(-2\beta q) \tag{17}$$

Suppose now that

$$\epsilon(p)=\sum_{k=1}^{p}kJ(k)=B(p)\log p \tag{18}$$

where $B(p)$ is monotone decreasing and such that $\epsilon(p)\to\infty$ as $p\to\infty$. Then by the monotonicity of $B(p)$

$$\epsilon(pM)-\epsilon(p)\leqslant B(p)\log M \tag{19}$$

Also,

$$\exp\left[2\beta\epsilon(p)\right]=p^{2\beta B(p)}$$

so using (16), (17), and (19),

$$Q \leqslant (2/M)(2p)^{2\beta B(2p)} \exp\left[2M\beta p^{-2\beta B(p)} B(p) \log M\right]$$

For any *fixed* β, setting

$$M = p^{2\beta B(p)} L(\beta)$$

gives

$$\begin{aligned} Q &\leqslant \frac{1}{L} 2^{[2\beta B(p)+1]} \exp\left[2\beta L B(p)(2\beta B(p) \log p + \log L)\right] \\ &= \frac{1}{L} 2^{[2\beta B(p)+1]} \exp\left[2\beta B(p) L \log L\right] \exp\left[4\beta^2 B(p)^2 L \log p\right] \end{aligned}$$

Suppose now that

$$B(p) = C(p)(\log p)^{-1/2} \tag{20}$$

with $C(p) \to 0$ as $p \to \infty$. Then, it is easily seen from (18) and (20) that $Q \leq 2/L$ as $p \to \infty$ provided

$$\lim_{N\to\infty} \left[(\log N)^{-1/2} \sum_{k=1}^{N} kJ(k) \right] = 0 \tag{21}$$

Recall that $N = Mp = p^{1+2\beta B(p)} L(\beta)$, so $N \to \infty$ as $p \to \infty$. It follows then from the arbitrariness of L that $Q \to 0$ as $N \to \infty$. Thus there is no long-range order at any temperature in the one-dimensional Ising model if (21) is satisfied.

3.2. Plane Rotator Model

For $n = 2$, (3) reduces to the plane rotator model for which[4]

$$V(q) = \frac{I_1(q)}{I_0(q)}$$

where I_μ is the modified Bessel function of order μ. From the asymptotic behavior of I_μ,[5] we have for sufficiently large q

$$\frac{3}{8\beta q} \leqslant 1 - V(q) = W(q) \leqslant \frac{5}{8\beta q} \tag{22}$$

Suppose as before that

$$\epsilon(p) = B(p) \log p \tag{23}$$

where $B(p)$ is monotone decreasing and such that $\epsilon(p) \to \infty$ as $p \to \infty$. Then by the monotonicity of $B(p)$

$$\epsilon(pM) - \epsilon(p) \leqslant B(p) \log M \tag{24}$$

Setting

$$M = B(p)\log(p)L(\beta)$$

and using (16), (22), (23), (24) we obtain

$$\begin{aligned} Q &\leqslant \frac{16\beta\log 2}{3L}\exp\left[\frac{5}{8}LB(p)\log M\right] \\ &\leqslant \frac{5\beta}{L}\exp\left\{LB(p)\left[\log L + \log B(p)\right]\right\}\exp\left[LB(p)\log\log p\right] \end{aligned}$$

Suppose

$$B(p) = C(p)(\log\log p)^{-1} \tag{25}$$

where $C(p) \to 0$ as $p \to \infty$. Then, as L is arbitrary, (23) and (25) show $Q \to 0$ as $p \to \infty$ provided

$$\lim_{N\to\infty}\left[\frac{\log\log N}{\log N}\sum_{k=1}^{N} kJ(k)\right] = 0 \tag{26}$$

The same argument as above shows there is no long-range order at any temperature in the one-dimensional plane rotator model if (26) is satisfied.

4. DISCUSSION

Comparison of (21) and (26) indicates that the nonexistence of long-range order has been established for a wider class of interactions in the plane rotator than the Ising model. This is to be expected in view of the comparison equality[6]

$$\langle \mathbf{S}_k \cdot \mathbf{S}_i \rangle_{\text{plane rotator}} \leqslant \langle \mu_k \mu_i \rangle_{\text{Ising}}$$

Indeed for the interaction $J(n) = n^{-2}$, it seems likely from our results that the plane rotator has no long-range order, while the Ising model has a phase transition with long-range order at sufficiently low temperatures.

Additional comparison inequalities can be used to extend our results. For example, when $n = 1$, Griffiths' comparison inequality (see Ref. 7) shows that our conclusion is also valid for *nonferromagnetic* pair interactions satisfying

$$\sum_{k=1}^{N} k|J(k)| = o(\log N)^{1/2} \qquad \text{as } N \to \infty$$

Also when $n > 2$, Bricmont's comparison inequalities[8] show in particular that the mean square magnetization is bounded above by its $n = 2$ value. It follows that there is zero long-range order for the n-vector model provided (26) holds and $n \geqslant 2$.

Finally, we point out that from a result of Lebowitz[9] relating zero boundary conditions correlation functions and plus boundary conditions correlation functions for $n = 1$, our results also imply zero spontaneous magnetization when (21) holds and $n = 1$.

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APPENDIX: SPIN LOCKING IN THE PLANE ROTATOR MODEL

For the plane rotator we introduce the parametrization

$$\mathbf{S}_i = (\cos\theta_i, \sin\theta_i), \qquad -\pi < \theta_i \leqslant \pi$$

Then (4) becomes

$$\langle f(\mathbf{S})\rangle = Z_N^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(\mathbf{S})\exp(-\beta H)\, d\theta_1\cdots d\theta_N$$

Let i, j, and $r < N$ be sites in Λ. Then, by Laplace's method for asymptotic expansions of integrals, as $J_{r,r+1}\to\infty$

$$Z_N = \left(\frac{2\pi}{\beta J_{r,r+1}}\right)^{1/2}\left[\int_{-\pi}^{\pi}\cdots\int\left[\exp(-\beta H^*)\right]\Big|_{\theta_r=\theta_{r+1}} d\theta_1\cdots d\theta_r \times d\theta_{r+2}\cdots d\theta_N + O\left(\frac{1}{J_{r,r+1}}\right)\right]$$

and

$$Z_n\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle = \left(\frac{2\pi}{\beta J_{r,r+1}}\right)^{1/2}\left[\int_{-\pi}^{\pi}\cdots\int\left[\cos(\theta_j-\theta_i)\exp(-\beta H^*)\right]\Big|_{\theta_r=\theta_{r+1}} \times d\theta_1\cdots d\theta_r\, d\theta_{r+2}\cdots d\theta_N + O\left(\frac{1}{J_{r,r+1}}\right)\right]$$

where H^* is the same as in Section 1. The arguments of Section 1 are then repeated.

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